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A Simple SOCP formulation of Minimization of Network Congestion Ratio

Bimal Chandra Das, Eiji Oki and Masakazu Muramatsu

Department of Communication Engineering and Informatics
The University of Electro-Communications, Tokyo
E-mail: bcdasdiu@gmail.com, eiji.oki@uec.ac.jp, MasakazuMuramatsu@uec.ac.jp.

1 Introduction

The network link utilization rate is the ratio of traffic flow through a link and its capacity. The network congestion ratio is the maximum value of all links utilization rates. When some links or nodes in network broadcast too much information, it causes network congestion and decreases the network performance by taking more time to send information from source to destination, packet loss or reducing the throughput ([1]). How to minimize the congestion ratio is an important affairs of information and communication technology (ICT) sector.

There are several researches on the minimization of the network congestion ratio. Wang and Wang [2] proposed a specific routing problem for internet traffic engineering as a linear programming (LP) problem whose objective is to minimize the congestion ratio. In their work they assuming that the traffic demand d_{pq} for every pair (p, q) of source node p and destination node q is exactly known. In fact, due to the exact traffic matrix, $T = \{d_{pq}\}$ this research provides a acceptable routing performance than the Multi-Protocol Label Switching (MPLS) standard. The traffic model which is exploits the exact traffic matrix is known as the pipe model ([3], [4], [5]).

In this research, we propose a model based on robust optimization to minimize the network congestion ratio. We consider some fluctuations in the estimated traffic demands of the pipe model depending on a parameter applying robust optimization technique to build up second-order cone constraints and finally formulate the problem as the SOCP model.

2 Backbone Network and Network Model

The network is represented as a directed graph $G(V, A)$, where V is the set of vertices (nodes) and A is the set of links. Let $Q \subseteq V$ be the set of edge nodes through which traffic is admitted into and going outside the network. A link from $i \in V$ to node $j \in V \setminus \{i\}$ is denoted as $(i, j) \in A, i \neq j$. Let an edge node pair of $p \in Q$ and $q \in Q$, where $p \neq q$, be denoted by $(p, q) \in W$, where W is the set of edge-node pairs (p, q). x_{ij}^{pq} is the portion of traffic from node $p \in Q$ to node $q \in Q$ routed through link $(i, j) \in A$. c_{ij} is the capacity of link $(i, j) \in A$. The network congestion ratio, which refers to the maximum value of all link utilization rates in the network, is denoted by r. The admissible traffic in the network can be maximized by minimizing the network congestion ratio, r. The admissible traffic is accepted up to the current traffic volume multiplied by $1/r$.

The backbone network or network that interconnects various pieces of network provides paths for the exchange of information between different LANs or subnetworks. It can tie together diverse networks in the same area, in different areas, or over wide areas. A large corporation that has many locations may have a backbone network that ties all of the locations together. The network congestion ratio is often taken into consideration while designing the network. To minimize the network congestion ratio with routing control is the objective of this paper.

3 Pipe Model

In the pipe model, the traffic demand information $T = \{d_{pq} : (p, q) \in W\}$ are assumed to be exactly routed between source and destination nodes. The routing

Table 1: Summary of Notations

Parameters	Description
$G(V, A)$	Directed graph G with $\|V\|$ nodes and $\|A\|$ links
Q	Set of edge nodes, $Q \subseteq V$
W	Set of edge nodes pair of $p \in Q$ and $q \in Q, p \neq q$
$\bar{d}_{pq}$	Estimated traffic demand from node p to q
c_{ij}	Capacity of link $(i, j) \in A$
ϵ	Total error in the traffic demands
Variables	**Description**
x_{ij}^{pq}	Portion of traffic from node $p \in Q$ to $q \in Q \setminus \{p\}$ routed through link $(i, j) \in A$
d_{pq}	Traffic demand from node p to q
r	Networks congestion ratio

formulation for the pipe model to minimize the network congestion ratio, is as follows:

$$\min r \tag{1a}$$

$$\text{s.t.} \sum_{j:(i,j)\in A} x_{ij}^{pq} - \sum_{j:(j,i)\in A} x_{ji}^{pq} = 1, \quad \forall (p,q) \in W, i = p \tag{1b}$$

$$\sum_{j:(i,j)\in A} x_{ij}^{pq} - \sum_{j:(j,i)\in A} x_{ji}^{pq} = 0, \quad \forall (p,q) \in W, \forall i \in V \setminus \{p,q\} \tag{1c}$$

$$\sum_{(p,q)\in W} d_{pq} x_{ij}^{pq} \leq c_{ij} \cdot r, \forall (i,j) \in A \tag{1d}$$

$$0 \leq x_{ij}^{pq} \leq 1, \forall (p,q) \in W, \forall (i,j) \in A \tag{1e}$$

$$0 \leq r \leq 1. \tag{1f}$$

Here the constraints (1b) and (1c) are flow conservation constraints. Constraint (1b) represents that the total portion of traffic flow outgoing from node $i (= p)$ is equal to 1 and constraint (1c) states that the total portion of traffic incoming to node i must be same as the total portion of outgoing from node i if the node i is neither a source or destination node for the traffic flow. Constraint (1d) indicates that the sum of the portion of traffic demands broadcasted through the link (i, j) is equal to or less than the capacity of that link times the network congestion ratio. The objective function represented by Eq. (1a) minimizes the network congestion ratio. The pipe model generally achieves a high routing performance; however, it requires exact data of traffic demands T, which is sometimes difficult to obtain in reality.

4 Robust Optimization

When some coefficients of an optimization problem has uncertainty and we want to optimize the problem in the worst case with respect to the uncertainty in some sense, the resulting optimization problem is called a robust optimization problem ([7], [13], [8]).

In this paper, we propose a different type of assumptions on errors. Our model can capture deviations of demand to all over the network.

Specifically, we propose to bound the total amount of squared errors in $\bar{d}_{pq}$ for all $(p, q) \in W$ by a positive constant, ϵ, and the true demand is contained in:

$$\Theta_\epsilon = \left\{ \mathbf{d} : \sqrt{\sum_{(p,q)\in W} (d_{pq} - \bar{d}_{pq})^2} \leq \epsilon \right\}, \tag{2}$$

In our model, ϵ is a single network-wide parameter. It is easy to see the following inclusion relationships between the two sets, so we omit the proof. In Section 5, we propose a robust optimization model based on the error (2) to the pipe model.

Note that in our model, we need the *estimated value* denoted by $\bar{d}_{pq}$ for every $(p, q) \in W$ explicitly.

5 Robust Optimization Model for Pipe Model

We apply the robust optimization technique to the pipe model. Since the constraint $(1d)$ should be satisfied for every $\mathbf{d} \in \Theta_\epsilon$, we have the following inequality for each $(i, j) \in A$:

$$\max_{\mathbf{d}\in\Theta_\epsilon} \left(\sum_{(p,q)\in W} d_{pq} x_{ij}^{pq} \right) \leq c_{ij} \cdot r. \tag{3}$$

Now we evaluate the left hand side of (3) to obtain a second-order cone constraint. The following lemma plays a crucial role in evaluating the left hand side of (3).

Lemma 1 *Let* $\Omega_\theta = \{\boldsymbol{x} \in \mathbb{R}^n : ||\boldsymbol{x}|| \leq \theta\}$, *where* $||\cdot||$ *is the Euclidean norm. For given* $\boldsymbol{a} \in \mathbb{R}^n$ *and* $\theta > 0$, *we have*

$$\max_{\boldsymbol{x}\in\Omega_\theta} \boldsymbol{a}^T \boldsymbol{x} = \theta ||\boldsymbol{a}||.$$

Proof The Lagrangian function of the optimization problem,

$$\max_{\mathbf{x}\in\Omega_\theta} \mathbf{a}^T\mathbf{x}, \text{ s.t } ||\mathbf{x}|| \leq \theta, \ \forall \mathbf{x} \in \Omega_\theta \text{ is}$$

$$F(\mathbf{x}, \lambda) \equiv \mathbf{a}^T\mathbf{x} + \lambda(\theta - ||\mathbf{x}||).$$

Karush-Kuhn-Tucker (KKT) conditions at the optimal point are
(i) $\nabla_{\mathbf{x}} F(\mathbf{x}, \lambda) \equiv \mathbf{a} - \lambda \nabla_{\mathbf{x}} ||\mathbf{x}|| = 0$, (ii) $\lambda(\theta - ||\mathbf{x}||) = 0$,
(iii) $\theta - ||\mathbf{x}|| \geq 0$, (iv) $\lambda \geq 0$ From condition (i), we can write

$$\mathbf{a} - \lambda \nabla_{\mathbf{x}} ||\mathbf{x}|| = 0 \Leftrightarrow \mathbf{a} - \lambda \frac{\mathbf{x}}{||\mathbf{x}||} = 0 \Leftrightarrow \mathbf{a}||\mathbf{x}|| = \lambda \mathbf{x} \Leftrightarrow \mathbf{a}\theta = \lambda \mathbf{x}$$

$\Leftrightarrow ||\mathbf{a}||\theta = \lambda ||\mathbf{x}|| \Leftrightarrow ||\mathbf{a}||\theta = \lambda\theta \quad \therefore \lambda = ||\mathbf{a}||$
Again, $\mathbf{a}\theta = \lambda \mathbf{x} \quad \Leftrightarrow \mathbf{x} = \theta.\frac{\mathbf{a}}{||\mathbf{a}||} \Leftrightarrow \mathbf{a}^T\mathbf{x} = \theta.\frac{\mathbf{a}^T\mathbf{a}}{||\mathbf{a}||}$
$\Leftrightarrow \mathbf{a}^T\mathbf{x} = \theta.\frac{||\mathbf{a}||^2}{||\mathbf{a}||} \Leftrightarrow \mathbf{a}^T\mathbf{x} = \theta||\mathbf{a}||$.
$\mathbf{x}$ is the optimal solution of the above optimization problem, which indicates that $\max_{\mathbf{x}\in\Omega_\theta} \mathbf{a}^T\mathbf{x} = \theta||\mathbf{a}||$.

To apply Lemma 1 to evaluate the left hand side of (3), we introduce a variable:

$$v_{pq} = d_{pq} - \bar{d}_{pq}$$

for each $(p, q) \in W$. Then we easily see that

$$\mathbf{d} \in \Theta_\epsilon \Leftrightarrow \mathbf{v} \in \Omega_\epsilon.$$

Therefore, using Lemma 1, we have for every $(i, j) \in A$

$$\begin{aligned}
&\max_{\mathbf{d}\in\Theta_\epsilon}\left(\sum_{(p,q)\in W} d_{pq}x_{ij}^{pq}\right) \\
&= \max_{\mathbf{v}\in\Omega_\epsilon}\left(\sum_{(p,q)\in W} v_{pq}x_{ij}^{pq}\right) + \sum_{(p,q)\in W} \bar{d}_{pq}x_{ij}^{pq} \\
&= \epsilon\sqrt{\sum_{(p,q)\in W}\left(x_{ij}^{pq}\right)^2} + \sum_{(p,q)\in W} \bar{d}_{pq}x_{ij}^{pq}. \qquad (4)
\end{aligned}$$

Substituting the left hand side of (3) by (4), we obtain the equivalent inequality for every $(i, j) \in A$:

$$\sqrt{\sum_{(p,q)\in W}\left(x_{ij}^{pq}\right)^2} \leq \frac{1}{\epsilon}\left(c_{ij}\cdot r - \sum_{(p,q)\in W} \bar{d}_{pq}x_{ij}^{pq}\right).$$

We now introduce a second-order cone programming problem (SOCP).

The closed convex cone

$$\mathrm{SOC}(1 + r) = \left\{\mathbf{x} \in \mathbb{R}^{1+r} : x_0 \geq \sqrt{\sum_{j=1}^{r} x_j^2}\right\}$$

is called the $1+r$ dimensional *second-order cone*. When a subvector of variables is restricted in a suitable dimensional second-order cone, such a constraint is called a second-order constraint. If an optimization problem has only a linear objective function, linear constraints, and second-order cone constraint, then such an optimization problem is called a second-order cone programming problem (SOCP).

An SOCP can be solved very efficiently by the primal-dual interior-point methods ([11], [12]). In fact, the modern optimization softwares such as SCIP, CPLEX, or Gurobi ([9], [10],[6]) can handle SOCP.

The constraint in (8) containing square root can be casted into the following form using the second-order cone:

$$w_{pq}^{ij} = x_{ij}^{pq} \tag{5}$$

$$w_0^{ij} = \left(c_{ij} \cdot r - \sum_{(p,q)\in W} \bar{d}_{pq} x_{ij}^{pq} \right) / \epsilon \tag{6}$$

$$\begin{pmatrix} w_0^{ij} \\ \mathbf{w}^{ij} \end{pmatrix} \in \mathrm{SOC}(1 + |W|), \tag{7}$$

where $\mathbf{w}^{ij} = (w_{pq}^{ij})_{(p,q)\in W}$. The first two constraints are linear, and the last one is a second-order cone constraint. As a result, we obtain an SOCP as a robust optimization model of the pipe model as follows:

$$\begin{aligned} \min\ & r \\ \text{s.t.}\ & \text{Eqs. (1b), (1c)} \\ & \text{Eqs. (5), (6), (7),} \qquad (i,j) \in A \\ & \text{Eqs. (1e), (1f).} \end{aligned} \tag{8}$$

By introducing the SOCP model the operators can deal without knowing the exact traffic demand by allowing them to total error in the estimated traffic demand. We formulate the problems as second-order cone programming problems whose objective is to minimize the network congestion ratio in the case of traffic fluctuation. In SOCP model, we can make many fluctuations in the estimated traffic demand which is a major advantage of this model. Effectiveness of SOCP model compared to the others model to minimize the congestion is future work.

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